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Isobar Approximation of Production Processes

S. MANDELSTAM, J. E. PATON, RONALD F. PEIERLS,* AND A. Q. SARKER

Department of Mathematical Physics, University of Birmingham, England

The three-particle term in the unitarity relation is approximated by treating pairs of particles as isobars. The method is applied to investigate the (33) resonance contribution to inelastic π–N scattering and to the model for the second π–N resonance proposed by one of us.

I. INTRODUCTION

In the last few years considerable progress has been made in utilizing the properties of analyticity, unitarity, and Lorentz invariance in calculating the two-particle scattering amplitudes. Application of these principles to amplitudes for more than two particles is still a long way from being achieved in view of the complications arising due to the growing number of invariants and the fact that the analytic properties of such amplitudes are not yet known.

In the present work we introduce an approximation scheme to reduce the problem of three particles to an equivalent two-body problem. This is achieved by taking advantage of the observed resonances between different pairs of particles. The approximation scheme is quite general and as an example of its applications it is applied to the problem of pion–nucleon interaction above the inelastic threshold. Here it is assumed that the 33 resonance is in some sense basic and attempts are made to reproduce the higher resonances of pion–nucleon scattering in terms of the parameters of the 33 resonance. This is motivated primarily by the qualitative success of a similar approach to the problem in an earlier work (*1*).

In Section II we introduce the approximation scheme and explain its effect on the exact unitarity relation with three intermediate particles. This formalism is then applied to pion–nucleon scattering above the inelastic threshold in Section III. The amplitudes are written down in an invariant way. Due to complications arising from spin 3/2, we find it convenient to work with helicity states. In Section IV we apply the formalism to the mechanism for the second resonance proposed by one of us and give approximate solutions of the integral equations of

* Now at the Institute for Advanced Study, Princeton, N. J.

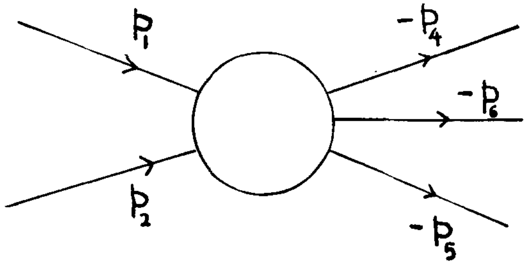

FIG. 1. The production of three particles from two.

the problem, and in Section V we discuss the results. In the Appendices we give derivations of the helicity amplitudes and discuss their isotopic spin dependence.

II. APPROXIMATION TO THE UNITARITY RELATION

In this section we give a precise meaning to the isobar model and shall show how, in this model, the three-particle unitarity term may be simplified. We neglect the spin complications. The results we get for spin zero particles and s-wave resonances have an obvious generalization which we assume to hold when spin is properly taken into account. Isotopic spin is also ignored for the time being.

A. THE ISOBAR MODEL

We consider the production of three spinless particles from two (Fig. 1):

$$1 + 2 \to 4 + 5 + 6. \tag{2.1}$$

Let us suppose that, of the final particles, 4 and 5 are capable of interacting through an s-wave resonance. Then one expects that the amplitude for this process has large contributions from those cases where 4 and 5 emerge in a relative s-state, and therefore emerge isotropically in their center-of-mass system. To describe the kinematics of reaction (2.1) requires five independent scalar invariants. Let p_1, p_2, $-p_4$, $-p_5$, $-p_6$ be the four-momenta of the particles: we define

$$s_{ij} = (p_i + p_j)^2. \tag{2.2}$$

Then we may choose s_{12}, s_{14}, s_{16}, s_{45}, s_{24} as our five invariants. Now we consider the center-of-mass frame of particles 4 and 5 (Fig. 2). s_{45} is the total energy of the particles 4 and 5; s_{12} and s_{11} (together with the masses of the particles and s_{45}) determine the configuration of the momenta p_1, p_2, p_6, and s_{14}, s_{24} determine the direction of p_4 (or p_5). Accordingly, that part of the amplitude for which 4 and 5 are in a relative s-state, is independent of s_{14} and s_{24}. Further, one expects this contribution to have a resonant dependence on s_{45} with position and width corresponding to the parameters of the resonance between 4 and 5.

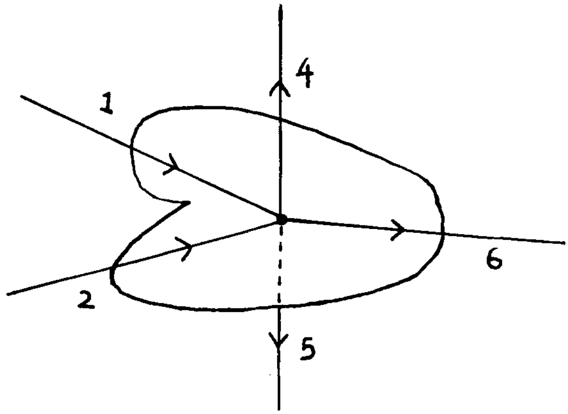

FIG. 2. Production process (2.1) in the center-of-mass system of 4 and 5.

Hence, if we can approximate the low-energy elastic scattering of 4 and 5 by a one-level formula:

$$\langle 5'4' \mid \mathbf{T} \mid 45 \rangle = \frac{\beta}{M^2 - i\Delta - s_I}$$
$$s_I = s_{45}, \qquad \beta = \frac{8\pi M\Delta}{q_r} \tag{2.3}$$

where M is the total center-of-mass energy, Δ the width of the resonance, and q_r the corresponding momentum, we can write the amplitude for the process (2.1) as

$$\langle 654 \mid \mathbf{T} \mid 12 \rangle = F(s, t, s_I) \frac{\beta^{1/2}}{M^2 - i\Delta - s_I} + f_0(s_{12}, \cdots, s_{45})$$
$$s = s_{12}, \qquad t = s_{16}. \tag{2.4}$$

For a narrow resonance $F(s, t, s_I)$ does not vary much with s_I and may, therefore, be replaced by a mean value $F(s, t, M^2)$, which depends only on the two kinematic variables s and t and is similar to an ordinary two-body scattering amplitude. We call it the amplitude for "isobar" production and by the isobar model we mean the assumption that the inelastic scattering is dominated by the isobar production term, and that the term $f_0(s_{12}, \cdots, s_{45})$ is in general small.

In the isobar model as formulated by Sternheimer and Lindenbaum (*2*) it is assumed in addition that $F(s, t, M^2)$ is more or less independent of t, so that the cross sections may be determined by phase space modified by the resonance factor (2.3). We shall show how unitarity leads to equations coupling $F(s, t, M^2)$ to the elastic scattering amplitude and to other isobar scattering amplitudes. These make it possible in principle to calculate $F(s, t, M^2)$.

We can take into account similar resonances between particles (6, 5) and (6, 4) by approximating the amplitude for reaction (2.1) by

$$\langle 654 \mid \mathbf{T} \mid 12 \rangle = \sum_{k=1}^{3} F_k(s_k, t_k, M_k^2) \frac{\beta_k^{1/2}}{M_k^2 - i\Delta_k - s_k} + f_0(s_{12}, \cdots, s_{45}). \quad (2.5)$$

As before the term $F_0(s_{12}, \cdots, s_{45})$ in (2.5) is small and will be dropped.

We are interested in the three-particle intermediate states. By unitarity, the imaginary part of the amplitude for the elastic scattering

$$1 + 2 \rightarrow 1' + 2' \quad (2.6)$$

will include a term of the general form

$$\int \langle 1' 2' \mid \mathbf{T} \mid 456 \rangle^* \langle 654 \mid \mathbf{T} \mid 12 \rangle \, d\tau \quad (2.7)$$

where the integration is taken over the phase space of particles 4, 5, and 6. Inserting (2.5) will involve four types of term

$$\int F_k'^* F_k \, d\tau \quad (2.8)$$

$$\int F_k'^* F_l \, d\tau_{kl} \quad (2.9)$$

$$\int (F_k'^* F_0 + F_0'^* F_k) \, d\tau_k \quad (2.10)$$

$$\int F_0'^* F_0 \, d\tau \quad (2.11)$$

where we have defined certain regions of phase space

$$\tau_k : \quad (s_k - M_k^2)^2 \lesssim \Delta_k^2$$
$$\tau_{kl} : \quad (s_k - M_k^2)^2 \lesssim \Delta_k^2, \qquad (s_i - M_i^2)^2 \lesssim \Delta_i^2.$$

Our approximation will be to keep only terms of type (2.8). In other words we assume

(a) $$\int F_k'^* F_k \, d\tau_k \gg \int F_k'^* F_0 \, d\tau_k \gg \int F_0'^* F_0 \, d\tau$$

and (2.12)

(b) $$\int d\tau_{kl} \ll \int d\tau_k .$$

The condition (b) states that the region of phase space where two isobars can simultaneously be formed is very small.

B. The Unitarity Relation

We define the T-matrix in terms of the S-matrix by

$$\langle f \mid \mathcal{S} \mid i \rangle = \langle f \mid i \rangle + i(2\pi)^4 \delta(f - i) \langle f \mid \mathfrak{J} \mid i \rangle \tag{2.13}$$

where i is the initial and f the final state and $\delta(f - i)$ ensures four-momentum conservation. If i and f are both three-particle states then we can separate out from $\langle f \mid \mathfrak{J} \mid i \rangle$ a number of terms which depend on one initial and one final momentum through a δ-function only. Assuming the particles to be distinct, we can express the three-particle $\mathfrak{J}$-matrix as

$$\begin{aligned}\langle 654 \mid \mathfrak{J} \mid 123 \rangle = \delta_{14} \langle 65 \mid \mathfrak{J} \mid 23 \rangle + \delta_{36} \langle 54 \mid \mathfrak{J} \mid 12 \rangle \\ + \delta_{25} \langle 64 \mid \mathfrak{J} \mid 13 \rangle + \langle 654 \mid \mathfrak{J}' \mid 123 \rangle .\end{aligned} \tag{2.14}$$

The invariant amplitudes $\mathbf{T}$ are defined in terms of the matrix elements of $\mathfrak{J}$ in the usual way,

$$\begin{aligned}\langle 54 \mid \mathfrak{J} \mid 12 \rangle &= (16\omega_1\omega_2\omega_4\omega_5)^{-1/2} \langle 54 \mid \mathbf{T} \mid 12 \rangle \\ \langle 654 \mid \mathfrak{J} \mid 12 \rangle &= (32\omega_1\omega_2\omega_4\omega_5\omega_6)^{-1/2} \langle 654 \mid \mathbf{T} \mid 12 \rangle \\ \langle 654 \mid \mathfrak{J}' \mid 123 \rangle &= (64\omega_1\omega_2\omega_3\omega_4\omega_5\omega_6)^{-1/2} \langle 654 \mid \mathbf{T} \mid 123 \rangle\end{aligned} \tag{2.15}$$

etc.,

$$\omega_i = (p_i^2 + \mu_i^2)^{1/2}$$

where μ_i is the mass of the particle.

If we confine our attention to the production of one particle only then the three processes

$$1 + 2 \rightarrow 4 + 5 \tag{2.16}$$

$$1 + 2 \rightarrow 4 + 5 + 6 \tag{2.17}$$

$$1 + 2 + 3 \rightarrow 4 + 5 + 6 \tag{2.18}$$

are coupled by unitarity. For definiteness we take particles one and 4 as spinless nucleons of mass m and particles 2, 3, 5, 6 as pions of mass μ. This simplifies calculations greatly but does not change any of the essential features of our approximations. Then the unitarity relation for the production process (2.17),

$$\sum_n \langle 654 \mid \mathcal{S}^+ \mid n \rangle \langle n \mid \mathcal{S} \mid 12 \rangle = 0$$

yields

$$2\ \mathrm{Im}\ \langle 654 \mid \mathbf{T} \mid 12\rangle = (\mathrm{a}) + (\mathrm{b}) + (\mathrm{c}) + (\mathrm{d}) + (\mathrm{e}), \tag{2.19}$$

where

$$(\mathrm{e}) = \int \frac{d^4p_7}{(2\pi)^4} \cdot \frac{d^4p_8}{(2\pi)^4} \cdot 2\pi\delta(p_7^{\,2} - m^2) \cdot 2\pi\delta(p_8^{\,2} - u^2) \cdot (2\pi)^4 \cdot \delta^4(p_1 + p_2 + p_5 - p_7 - p_8) \tag{2.20}$$

$$\cdot \delta^4(p_7 + p_8 + p_4 + p_6) \times \theta(\omega_7)\theta(\omega_8) \langle 46 | \mathbf{T} \mid 78\rangle^* \langle 785 \mid \mathbf{T} \mid 12\rangle$$

and (a)–(d) can be written down analogously from the corresponding diagrams in Fig. 3. Terms (a) and (b) couple the inelastic amplitude with the elastic two-particle and three-particle amplitudes respectively. The remaining three terms may be regarded as final state interaction terms.

As discussed above we approximate the five-legged and six-legged diagrams by isobar production and isobar scattering terms. In this case one of the isobars is a pion–pion isobar of mass $\mathfrak{m}$, say, and is denoted by a wavy line. The other two isobars are pion-nucleon isobars and will be denoted by a double line. We write

$$\langle 654 \mid \mathbf{T} \mid 12\rangle = \sum_{ab} F(ab, c; 12) g(a, b) \tag{2.21}$$

summed over $ab = 45, 56, 46$;

$$\langle 654 \mid \mathbf{T} \mid 123\rangle = \sum_{ab} \sum_{de} F(ab, c; de, f) g(a, b) g(d, e) \tag{2.22}$$

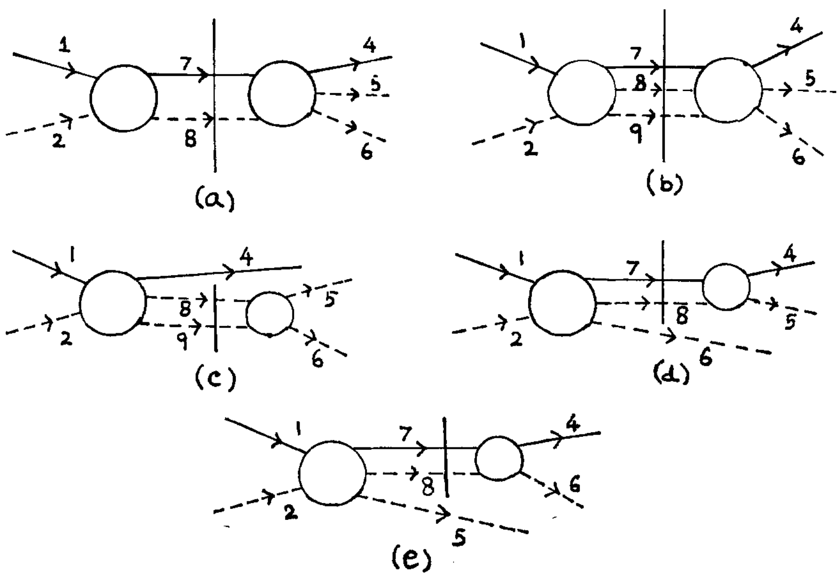

FIG. 3. Diagrammatic representation of right-hand side of Eq. (2.20). The vertical lines indicate the intermediate states summed over.

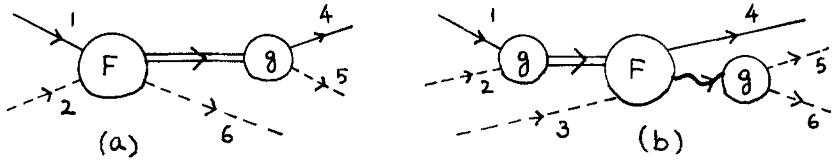

FIG. 4. Isobar contributions to three-particle amplitudes.

summed over $ab = 45, 56, 46$; $de = 12, 23, 13$ (c and f in this and (2.21) being the remaining member of the triad), where

$$g(a, b) = \frac{\beta^{1/2}}{(p_{ab}^2 - m_{ab}^2 + i\Delta_{ab})}; \qquad p_{ab} = (p_a + p_b) \tag{2.22a}$$

and m_{ab}, Δ_{ab} are the mass and width of the appropriate isobar. Figure 4 gives a graphical representation of the two terms $F(45, 6; 12)g(45)$ and $F(4, 56; 12, 3)g(12)g(56)$, contributing to (2.21) and (2.22) respectively.

We now substitute (2.21) and (2.22) into (2.20) and identify the isobar contributions on both sides of the equation. We then get an equation for the imaginary part of each of the terms on the right-hand side of (2.21). We consider in detail only one term, say $2 \operatorname{Im} g(56)F(4, 56; 1, 2)$. Diagrammatically the right-hand side is represented by Fig. 5 (obtained by inserting all diagrams of the type illustrated in Fig. 4 into the appropriate part of Fig. 3).

All these diagrams have in common the final state pion–pion interaction. The integration occurring in the diagram of Fig. 5(a) can be carried out and it contributes a term which is canceled out by the corresponding term in $2 \operatorname{Im} g(65)$ $F(65, 4; 1, 2)$ on the left. Within our approximations (conditions (2.12) (a), (b)) the contributions from diagrams of Fig. 5 (b), (d), (e) and (f) may be neglected. Finally the integrations in (g) and (h) reduce to the two-particle unitarity integral with appropriate factors of $(2\pi)^{-2}$, taking into account the identity of the two pions. Thus we obtain

$$\begin{aligned} 2 \operatorname{Im} F(65, 4; 1, 2) = {} & \frac{1}{(2\pi)^2} \int d^4p_1\, d^4p_8\, F(65, 4; 7, 8)h^*(8, 7; 1, 2) \\ & \times \theta(\omega_7)\theta(\omega_8)\delta(p_7^{\,2} - m^2)\delta(p_8^{\,2} - \mu^2) \\ & + \frac{1}{(2\pi)^2} \int d^4p_{78}\, d^4p_9\, F(65, 4; 78, 9)F^*(9, 78; 1, 2) \\ & \times \theta(\omega_{78})\theta(\omega_9)\delta(p_{78}^2 - M^2)\delta(p_9^{\,2} - \mu^2) \\ & + \frac{1}{(2\pi)^2} \int d^4p_{98}\, d^4p_7\, F(65, 4; 7, 98)F^*(98, 7; 1, 2) \\ & \times \theta(\omega_{98})\theta(\omega_7)\delta(p_{98}^2 - \mathrm{m}^2)\delta(p_7^{\,2} - m^2) \end{aligned} \tag{2.23}$$

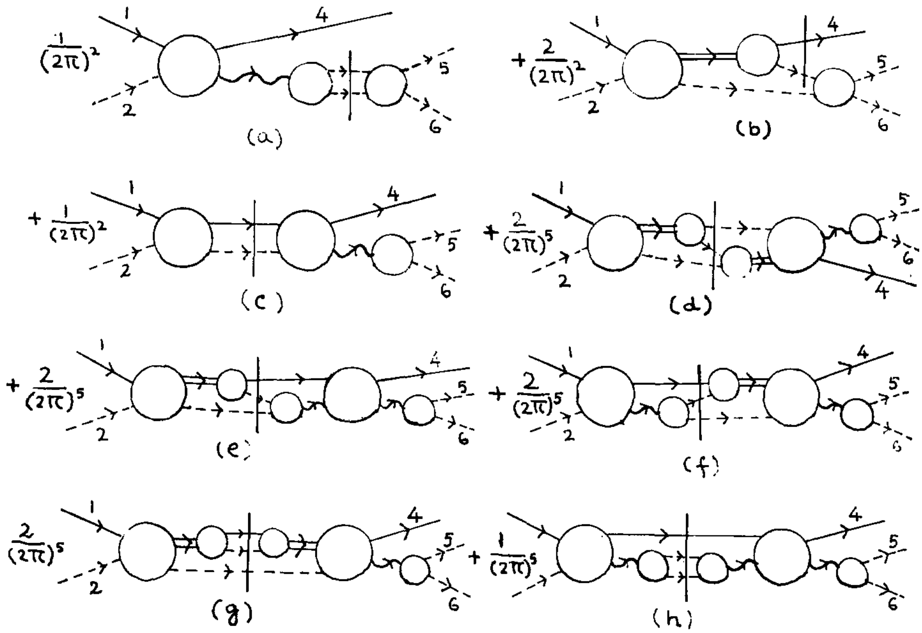

FIG. 5. Unitarity diagrams for the amplitude $g(56)F(4,56;1,2)$.

where $h(7, 8: 1, 2)$ is the π–N elastic amplitude. Similarly it can be shown that

$$
\begin{aligned}
2 \operatorname{Im} F(6, 45; 1, 2) &= \frac{1}{(2\pi)^2} \int d^4p_7\, d^4p_8\, F(6, 45; 7, 8)h^*(8, 7; 1, 2) \\
&\qquad \times \theta(\omega_7)\theta(\omega_8)\delta(p_7^2 - m^2)\delta(p_8^2 - \mu^2) \\
&+ \frac{1}{(2\pi)^2} \int d^4p_{78}\, d^4p_9\, F(6, 45; 78, 9)F^*(9, 78; 1, 2) \\
&\qquad \times \theta(\omega_{78})\theta(\omega_9)\delta(p_{78}^2 - M^2)\delta(p_9^2 - \mu^2) \\
&+ \frac{1}{(2\pi)^2} \int d^4p_{98}\, d^4p_7\, F(6, 45; 7, 98)F^*(98, 7; 12) \\
&\qquad \times \theta(\omega_{98})\theta(\omega_7)\delta(p_{98}^2 - m^2)\delta(p_7^2 - m^2)
\end{aligned} \tag{2.24}
$$

The last two equations are the usual unitarity relations for the isobar production amplitudes expressed in terms of themselves and the pion–nucleon and the pion–isobar scattering amplitudes.

We can also treat the three-body scattering process (2.18) in a similar way, except that its unitarity relation contains extra pole terms corresponding to the diagrams of Fig. 6.

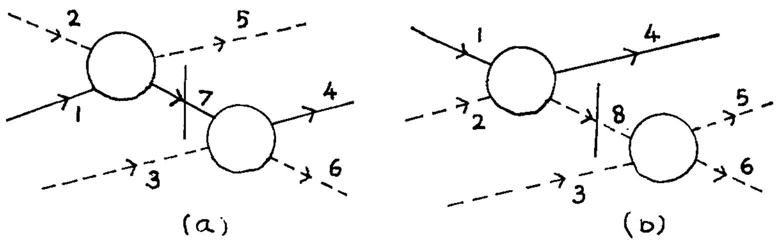

FIG. 6. Single-particle contributions to the unitarity relations for three-particle amplitudes.

It might seem at first sight that the contributions of Fig. 6 violate our condition (2.12) (b) on the nonoverlapping of isobar dominated phase space regions, which forbids Fig. 5(d), for example. However, the distinction can be seen from the fact that there is no integration involved over the pion variables in Fig. 6(a).

III. PION-NUCLEON INTERACTION ABOVE INELASTIC THRESHOLD

We now apply the approximation scheme of the last section to pion–nucleon scattering above the inelastic threshold. Attempts have been made to explain the various features of pion–nucleon cross sections in terms of the so-called "isobar model" (*2*, *3*). In this model it is assumed that the nucleon and one of the final state mesons always emerge in the relative resonant 33 state. If one considers the production of only one meson in the final state this configuration can account for the angular momenta and parities of the second and third resonances of pion–nucleon cross sections. Other analyses (*4*, *5*) also indicate that the final state described by the 33 isobar plays an important role in pion–nucleon interactions above the inelastic threshold. But so far there has been little quantitative discussion or explanation of the resonant peaks. In an earlier work (*1*) it has been suggested that such a simple model might give a quantitative description of the higher resonances. Therefore, ir is considered worthwhile to investigate the problem in detail.

Ball and Frazer (*6*) proposed a different explanation for the second resonance of pion–nucleon scattering in which the principal mechanism is the production of the pion–pion ($J = 1, I = 1$) resonance. In our approximation scheme there is scope for the inclusion of such resonances as well, but in the following considerations we shall not take into account the contributions of the pion–pion resonance.

We denote the 33 isobar by N^*. The processes we are interested in are (Fig. 7)

$$\pi + N \rightarrow \pi + N^* \tag{I}$$

$$\pi + N^* \rightarrow \pi + N^*. \tag{II}$$

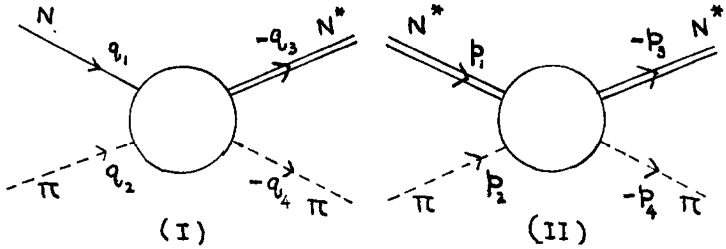

FIG. 7. Processes (I) and (II).

We write down some of the usual kinematical relations which we shall use very often.

$$\begin{aligned} s &= (q_1 + q_2)^2 = (q_3 + q_4)^2 \\ t &= (q_1 + q_3)^2 = (q_2 + q_4)^2 \\ u &= (q_1 + q_4)^2 = (q_2^2 + q_3)^2. \end{aligned} \tag{3.1}$$

In the center-of-mass system

$$\begin{aligned} s &= W^2 = (E_q + \omega_q)^2 = (E_q' + \omega_q')^2 \\ t &= m^2 + M^2 - 2E_qE_q' + 2qq' \cos\theta \\ u &= m^2 + \mu^2 - 2E_q\omega_q' - 2qq' \cos\theta \end{aligned} \tag{3.2}$$

where θ is the scattering angle and E_q, E_q' the energies of the incoming nucleon of mass m and the outgoing isobar of mass M, defined by

$$E_q = (q^2 + m^2)^{1/2}, \qquad E_q' = (q'^2 + M^2)^{1/2}. \tag{3.3}$$

q and q' are the incoming and outgoing momenta. The incoming and outgoing meson energies are

$$\omega_q = (q^2 + \mu^2)^{1/2}, \qquad \omega_q' = (q'^2 + \mu^2)^{1/2}. \tag{3.4}$$

Similar relations hold also for the process (II), in which case the magnitudes of the incoming and outgoing momenta in the center-of-mass system are the same and will be denoted by p.

A. SPIN DESCRIPTION OF THE ISOBAR

A convenient way of describing a spin $\frac{3}{2}$ particle is the Rarita and Schwinger (*9*) wave equation:

$$(\gamma \cdot k - M)\psi_{\lambda\alpha}(k) = 0 \tag{3.5}$$

where k is the four-momentum of the isobar and the wave function $\psi_{\lambda\alpha}(k)$ is a 16-component entity with λ as a four vector index and α as a four spinor index. It is subject to the subsidiary conditions

$$\begin{aligned} k\cdot\psi &= 0 \\ \gamma\cdot\psi &= 0 \end{aligned} \tag{3.6}$$

which cut down the number of independent components of ψ to eight. We multiply ψ by γ_5 in order that $\gamma_5\psi$ should describe a particle having the same transformation properties under reflection as the isobar.

We now calculate the coupling G of the isobar with a nucleon and a meson from the given parameters of the 33 resonance of pion–nucleon scattering. Near this resonance the pion–nucleon scattering amplitude has its sole contribution from the diagram of Fig. 8.

On denoting the nucleon spinor by $u(p)$ the amplitude A corresponding to Fig. 8 may be written as

$$\begin{aligned} A &= \frac{2MG^2 \sum \bar{u}(-p')q_\lambda'\psi_\lambda(K)\bar{\psi}_\mu(K)q_\mu\, u(p)}{K^2 - M^2 + i\Delta} \\ &= \frac{2MG^2\bar{u}(-p')q_\lambda'\mathcal{P}_{\lambda\mu}\, q_\mu\, u(p)}{K^2 - M^2 + i\Delta} \end{aligned} \tag{3.7}$$

where Δ is the width of the 33 resonance and $\mathcal{P}_{\lambda\mu}$ the projection operator given by

$$\mathcal{P}_{\lambda\mu} = \frac{1}{3}\left(3\delta_{\lambda\mu} - \frac{4K_\lambda K_\mu}{M^2} - \gamma_\lambda\gamma_\mu + \frac{\gamma_\lambda\gamma\cdot KK_\mu + K_\lambda\gamma\cdot K\gamma_\mu}{M^2}\right)\left(\frac{\gamma\cdot K + M}{2M}\right). \tag{3.8}$$

The expression (3.8) projects out only the positive energy state from the isobar wave function $\psi_{\lambda\alpha}(K)$. The amplitude A has a contribution only from the $J = 3/2$, $L = 1$ state. Comparing it with the standard one-level resonance formula for the 33 resonance we get

$$G^2 = \frac{24\pi M\Delta}{[(M + m)^2 - \mu^2]q_r^3}, \tag{3.9}$$

where q_r is the momentum corresponding to the 33 resonance.

B. Invariant Amplitudes

We now find out the independent invariant amplitudes which are necessary to describe the processes (I) and (II). For convenience we define the following

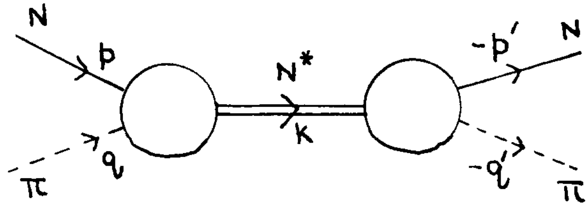

FIG. 8. Isobar approximation of pion–nucleon scattering.

quantities

$$\begin{aligned} Q &= \tfrac{1}{2}(q_2 - q_4) \\ K &= \tfrac{1}{2}(q_2 + q_4) \\ P &= \tfrac{1}{2}(q_1 - q_3) \end{aligned} \tag{3.10}$$

To begin with (I), we first observe that for a given angular momentum J, the possible transitions (labeled by initial and final orbital angular momentum) are $J \pm \tfrac{1}{2} \rightarrow J \pm \tfrac{1}{2}$, $J \mp \tfrac{3}{2}$, so that four amplitudes are required to describe the process (I). The amplitude $\langle \pi N^* \mid \mathbf{T} \mid \pi N \rangle$ has the structure of an invariant matrix $\mathfrak{M}_{\lambda,\alpha\beta}$ in spin space such that $\bar{\psi}_{\lambda\alpha}(q_\beta)\mathfrak{M}_{\lambda,\alpha\beta}u_\beta(q_1)$ is a spin scalar. A suitable choice[1] for $\mathfrak{M}_\lambda$ is

$$\mathfrak{M}_\lambda = (A_1 + \gamma \cdot QB_1)Q_\lambda + (A_2 + \gamma \cdot QB_2)K_\lambda \tag{3.11}$$

where the spinor indices are suppressed. The possibility of using γ, q_1 and q^2 or any other combination of them in Eq. (3.11) is ruled out by the Dirac equation and Eqs. (3.5) and (3.6). Apart from their isotopic spin dependence A_i and B_i are invariant scalar functions of the kinematic variables s, t, and u defined by Eq. (3.1).

Further, the matrix elements of $\mathfrak{M}_\lambda$ have certain analytic properties and we can show that the choice (3.11) introduces no additional kinematic singularities. This latter statement can be verified by taking traces of $\mathfrak{M}_\lambda$ between the subspaces spanned by nucleon and isobar wave functions and inverting the relation between the values of the traces and A_i and B_i. Four such possible traces are

$$T_1 = \frac{1}{2m} \operatorname{Tr}\{Q_\mu \, \mathcal{P}_{\mu\lambda} \, \mathfrak{M}_\lambda(\gamma \cdot q_1 + m)\}$$

$$T_2 = \frac{1}{2m} \operatorname{Tr}\,\{\gamma \cdot Q Q_\mu \, \mathcal{P}_{\mu\lambda} \, \mathfrak{M}_\lambda(\gamma \cdot q_1 + m)\}$$

$$T_3 = T_1(Q_\mu \rightarrow K_\mu), \qquad T_4 = T_2(Q_\mu \rightarrow K_\mu).$$

[1] This choice is similar to that of A. W. Hendry (Glasgow University, Department of Natural Philosophy Preprint).

The determinant from such an inversion has the value

$$(K^2)^4[(P\cdot Q)^2 - P^2Q^2]^4$$

which is zero only for forward and backward scattering. But the numerator of this inverted relation also has zeros for forward and backward scattering. Hence it is shown that no additional kinematic singularities are introduced.

We now consider the process (II). For a given value of the total angular momentum J the number of transitions from the initial π–N^* system to the final π–N^* system are

$$J \pm \tfrac{1}{2}, J \mp \tfrac{3}{2} \quad \leftrightarrow \quad J \pm \tfrac{1}{2}, J \mp \tfrac{3}{2}.$$

Among the eight possible transitions time reversal invariance implies that the amplitudes for the transitions $J + \frac{3}{2} \rightarrow J - \frac{1}{2}$, $J - \frac{1}{2} \rightarrow J + \frac{3}{2}$ are equal, and similarly those for the transitions $J + \frac{1}{2} \rightarrow J - \frac{3}{2}$ and $J - \frac{3}{2} \rightarrow J + \frac{1}{2}$ are the same. Hence the number of invariant amplitudes required to describe the process (II) is six.

As before [(3.10)], we can define the quantities Q, K, and P in terms of the four-momenta p_1, p_2, p_3, p_4 of the process (II) and express the invariant amplitude $\mathfrak{M}_{\lambda\mu}$ in terms of them. The possible choices are $\delta_{\lambda\mu}$, $Q_\lambda Q_\mu$, $K_\lambda K_\mu$, $Q_\lambda K_\mu - K_\lambda Q_\mu$ and $Q_\lambda K_\mu + K_\lambda Q_\mu$, of which the last one is ruled out by time reversal invariance. The remaining four can be shown to be related by

$$\delta_{\lambda\mu} = \frac{\begin{aligned}2K_\lambda K_\mu(P^2Q^2 - M\gamma\cdot QK^2) + 2Q_\lambda Q_\mu P^2K^2 \\ + (Q_\lambda K_\mu - K_\lambda Q_\mu)P^2(P\cdot Q - \gamma\cdot QM)\end{aligned}}{K^2(P^2Q^2 - (P\cdot Q)^2)} \tag{3.12}$$

and hence only three of them are independent. A suitable choice for $\mathfrak{M}_{\lambda\mu}$ is, therefore,

$$\begin{aligned}\mathfrak{M}_{\lambda\mu} = {} & (C_1 + \gamma\cdot QD_1)\delta_{\lambda\mu} \\ & + (C_2 + \gamma\cdot QD_2)Q_\lambda Q_\mu \\ & + (C_3 + \gamma\cdot QD_3)K_\lambda Q_\mu .\end{aligned} \tag{3.13}$$

We believe that such a choice introduces no additional kinematic singularities.

C. Helicity Amplitudes

We can now make the connection between the invariant amplitudes for processes (I) and (II) and the amplitudes in states of fixed total angular momentum. This may be done in two stages: First, relate the invariant amplitudes to the helicity amplitudes of Jacob and Wick (*8*); then do the partial wave projection of the helicity matrices. The method is the same as that used by

Goldberger *et al.* (*9*) for the nucleon–nucleon problem, and we give the algebraic details in Appendix A.

For process (I), define the normalization of the helicity matrix Φ by its connection with the center-of-mass cross section:

$$\frac{d\sigma}{d\Omega} = \frac{q'}{q} \mid \langle \lambda' \mid \mathbf{\Phi} \mid \lambda \rangle \mid^2 \tag{3.14}$$

where $\mathbf{\Phi}$ is a function of the center-of-mass variables W and θ, and λ, λ' are, respectively, the helicities of the incoming nucleon and outgoing isobar. Then define the helicity amplitudes ϕ_i and the corresponding ϕ_i^J in the state of total angular momentum J by

$$\begin{aligned}
\phi_1 &= \langle 3/2 \mid \mathbf{\Phi} \mid 1/2 \rangle = \frac{1}{2q} \sum_J (2J+1)\phi_1^J\, d^J_{1/2\,3/2}(\theta) \\
\phi_2 &= \langle 1/2 \mid \mathbf{\Phi} \mid 1/2 \rangle = \frac{1}{2q} \sum_J (2J+1)\phi_2^J\, d^J_{1/2\,1/2}(\theta) \\
\phi_3 &= \langle -1/2 \mid \mathbf{\Phi} \mid 1/2 \rangle = \frac{1}{2q} \sum_J (2J+1)\phi_3^J\, d^J_{1/2\,-1/2}(\theta) \\
\phi_4 &= \langle -3/2 \mid \mathbf{\Phi} \mid 1/2 \rangle = \frac{1}{2q} \sum_J (2J+1)\phi_4^J\, d^J_{1/2\,-3/2}(\theta)
\end{aligned} \tag{3.15}$$

The combinations of definite parity are:

$$\begin{aligned}
\phi^{J\pm}(1/2, 3/2) &= \phi_1^J \pm \phi_4^J \\
\phi^{J\pm}(1/2, 1/2) &= \phi_2^J \pm \phi_3^J
\end{aligned} \tag{3.16}$$

where the $(+)$ sign is associated with parity $(-1)^{J-1/2}$.

Define

$$S = \sin\theta; \qquad C = \cos\theta; \qquad \bar{s} = \sin\theta/2; \qquad \bar{c} = \cos\theta/2 \tag{3.17}$$

and

$$\begin{aligned}
G_1^{\pm} &= \frac{1}{8\pi W}\left[(F_1^+ \pm F_2^+) + (F_1^- \pm F_2^-)\right] \\
G_2^{\pm} &= \frac{1}{9\pi W}\left[(F_1^+ \pm F_2^+) - (F_1^- \pm F_2^-)\right]
\end{aligned} \tag{3.18}$$

where

$$F_i^{\pm} = \pm\sqrt{(E_q' \mp M)(E_q \pm m)}\left[A_i + \frac{M-m}{2} B_i \pm W B_i\right], i = 1, 2 \tag{3.19}$$

and denote by $G_n^{\pm}$ the following integrals, where $l = J - \frac{1}{2}$, $n = l, l \pm 1$, $l \pm 2$,

$$G_n^{\pm} = \int_{-1}^{1} dC(G_1^{\pm} + (-1)^{n-l}G_2^{\pm})P_n(C). \tag{3.20}$$

Then the combinations of (3.16) are given by

$$\begin{aligned}
\phi^{J+}(\tfrac{1}{2}, \tfrac{3}{2}) &= \frac{1}{2}\, l(l+2)q^2 \left[\frac{G_{l+2}^{+} - G_l^{+}}{2l+3} + \frac{G_{l+1}^{+} - G_{l-1}^{+}}{2l+1}\right] \\
\phi^{J+}(\tfrac{1}{2}, \tfrac{1}{2}) &= \frac{1}{\sqrt{6}}\left\{\frac{1}{2}\, q^2 \left[(l+2)\,\frac{G_{l+2}^{+} - G_l^{+}}{2l+3} + l\,\frac{G_{l+1}^{+} - G_{l-1}^{+}}{2l+1}\right]\right. \\
&\quad - \frac{E_q'}{M}\, q^2 \left[\frac{(l+2)G_{l+2}^{+} + (l+1)G_{l+1}^{+}}{2l+3} + \frac{(l+1)G_{l+1}^{+} + lG_{l-1}^{+}}{2l+1}\right] \\
&\quad \left. - \frac{1}{M}\, qq'[\omega_q(G_{l+1}^{+} + G_l^{+}) + W(G_{l+1}^{-} + G_l^{-})]\right\}
\end{aligned} \tag{3.21}$$

and $\phi^{J-}(\frac{1}{2}, \frac{3}{2})$, $\phi^{J-}(\frac{1}{2}, \frac{1}{2})$ may be obtained from the rule:

$$\text{Under} \qquad W \to -W, \qquad \begin{aligned} \phi^{J+}(\tfrac{1}{2}\ \tfrac{3}{2}) &\to \phi^{J-}(\tfrac{1}{2}\ \tfrac{3}{2}) \\ \phi^{J+}(\tfrac{1}{2}\ \tfrac{1}{2}) &\to -\phi^{J-}(\tfrac{1}{2}\ \tfrac{1}{2}). \end{aligned} \tag{3.22}$$

By analogy with (3.14) we can define the helicity matrix $\langle\lambda' \mid \zeta \mid \lambda\rangle$ for process (II), and ζ_i^J ($i = 1 \cdots 6$) the corresponding amplitudes in states of fixed angular momentum. This is done in Appendix A.

IV. APPLICATION TO THE SECOND RESONANCE IN PION-NUCLEON SCATTERING

In this section we shall apply the above formalism to investigate further the mechanism for the second resonance in pion–nucleon scattering proposed by one of us (*1*). Even though the result of an approximate calculation throws considerable doubt on the magnitude of the resonance effect produced by the mechanism we shall nevertheless present the calculation, since it involves features which are absent with elastic scattering and is, therefore, of some formal interest.

The mechanism for the resonance depended on the diagram, Fig. 9(a). The corresponding diagram before the isobar approximation is made (Fig. 9(b) contributes a term to the unitarity condition which was not present for the process $\pi + N \to \pi + N^*$ discussed in Section II, and which is not, in fact, in the isobar form. The transition amplitude F thus has an extra term

$$\frac{\beta^2}{(s_{12} - M^2 + i\Delta)(s_{45} - M^2 + i\Delta)(u - m^2 + i\epsilon)} \tag{4.1}$$

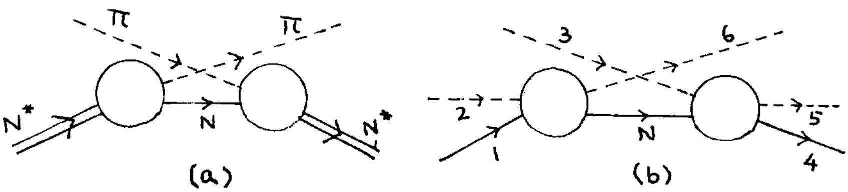

FIG. 9. Single-nucleon pole contribution to process (II).

where $u = (p_1 + p_2 - p_6)^2$. We shall neglect spin for simplicity. On using the usual relation between s, t and u, this becomes

$$-\frac{\beta^2}{(s_{12} - M^2 + i\Delta)(s_{45} - M^2 + i\Delta)(s + t - s_{12} - s_{45} - 2\mu^2 + m^2 - i\epsilon)}. \quad (4.2)$$

It is important to observe the $-i\epsilon$ instead of $+i\epsilon$ in the last denominator, owing to the reversal of sign in the relation between s and u. Thus the region of interest in the s-plane is below the cut due to the term (4.2), whereas it is above all other unitarity cuts. This squeezing of the amplitude between two cuts is a phenomenon which is known to occur for processes with six or more external lines (*10*).

The term (4.2) is not of the required resonance form, as it depends on s_{12} and s_{45} through the last denominator as well as the first two, and this is a strongly varying function in the region of interest. We shall therefore have to separate from it the "isobar" contributions. This will be done by observing that, in all the unitarity integrals, (4.2) will be multiplied by the complex conjugate of a scattering amplitude involving three particles. If it is of the isobar form, it will depend strongly on s_{12} or s_{45} only through factors of the form

$$(s_{12} - M^2 - i\Delta)^{-1} \quad \text{or} \quad (s_{45} - M^2 - i\Delta)^{-1}.$$

We must therefore find a function which, multiplied by these two factors and integrated over s_{12} and s_{45}, will give the same result as (4.2) multiplied by these factors and integrated over s_{12} and s_{45}. It must also depend on s_{12} and s_{45} only through the resonance denominators. Such a function is

$$-\frac{\beta^2}{(s_{12} - M^2 + i\Delta)(s_{45} - M^2 + i\Delta)(s + t - 2M - 2i\Delta - 2\mu^2 + m^2)} \quad (4.3)$$

so that, according to the theory of Section II, the expression corresponding to Fig. 9 (b) will be

$$-\frac{G^2}{(s + t - 2M^2 - 2i\Delta - 2\mu^2 + m^2)} \quad (4.4)$$

In other words, it could have been obtained by putting the squares of the masses of the external lines in Fig. 9(a) equal to $M^2 + i\Delta$, and calculating the diagram in the usual way. We shall assume that this prescription is also valid when spin is included. It is evident that the $2i\Delta$ in (4.4) must have a negative sign, corresponding to the negative sign before the $i\epsilon$ in (4.2), since the approximation must preserve the fact that the region of interest is below the singularity. The pole in (4.4) will thus be *in the physical sheet*, and will be a finite distance above the real axis. It will thus produce a spreading out due to the width of the pion–nucleon resonance.

One can now isolate a fixed partial wave, and the pole in (4.4) will become spread out into a short cut in the usual way; the cut is now in the upper half-plane on the physical sheet. If we further include intermediate pion–isobar states in the unitarity condition, but neglect pion–nucleon states to begin with, there will also be a cut on the real axis given by

$$d(F) = kF^*F. \tag{4.5}$$

The expression $d(F)$ represents the discontinuity in F, which is no longer equal to the imaginary part; k is a kinematic factor which depends on the spin. It is therefore necessary to find a function F with a known cut in the upper half-plane, and a cut given by (4.5) on the real axis. This is a problem which can be transformed by the N/D method to the solution of an integral equation.

We shall, however, make some simplifications so as to give an exactly soluble equation. First let us replace the cut in the upper half-plane by a pole which, with suitable choice of position and residue, should be possible without great loss of accuracy. Thus, the contribution of Fig. 9 to F will have the form

$$N(s) = \frac{r\Gamma}{k_0(s - N^2 - i\Gamma)} \tag{4.6}$$

where k_0 is the value of k at $s = N^2$. Further, we shall replace k in (4.5) by k_0, which should be accurate if the resonance is narrow. Thus

$$d(F) = 2ik_0F^*F. \tag{4.7}$$

The problem of finding a function (F) with a pole given by (4.6) and a cut along the real axis given by (4.7) can now be solved exactly. The result is

$$F(s) = N(s)/D(s) \tag{4.8}$$

where

$$\begin{aligned} D(s) &= 1 + \frac{2r}{(1 - 4r)^{1/2} + 1}\,\frac{s - N^2 - i\Gamma}{s - N^2 + i\Gamma}, \qquad \operatorname{Im} s > 0 \\ &= 1 + \frac{2r}{(1 - 4r)^{1/2} + 1}\,\frac{s - N^2 - i\Gamma}{s - N^2 - i\Gamma(1 - 4r)^{1/2}}, \qquad \operatorname{Im} s < 0. \end{aligned} \tag{4.9}$$

This solution is only valid if $r < \frac{1}{4}$, but it may easily be checked that (4.6) and (4.7) are incompatible above this limit. Our isobar model must therefore break down before r reaches the value $\frac{1}{4}$.

If we now turn to the reaction $\pi + N \to \pi + N^*$, but still keep only pion–isobar intermediate states, we obtain an Omnes-type equation. With the approximation that D is constant except near $S = N^2$, which is accurate if the resonance is narrow, the effect of the pion–isobar intermediate state is to multiply the other contributions to this amplitude by a factor

$$B(s) = \frac{(1 - 4r)^{1/2} + 1 + 2r}{(1 - 4r)^{1/2} + 1} D^{-1}. \tag{4.10}$$

The constant factor in (4.9) has been chosen to make B equal to 1 when S is far from N^2.

We may observe that the first expression for D in (4.9) has a zero when

$$s = N^2 - i\Gamma \frac{(1 - 4r)^{1/2} + 1 + 2r}{(1 - 4r)^{1/2} + 1 - 2r}. \tag{4.11}$$

This is on the unphysical sheet and therefore indicates the presence of a *true resonance*. However, if the value of r in (4.9) is much below $\frac{1}{4}$, $B(s)$ will not be very large at $s = N^2$, so that unless the amplitude for the reaction $\pi + N \to \pi + N^*$ was large to begin with, the effect will not be very great. Of course, we have consistently neglected the π–N intermediate state in the calculations. This state could be included at the cost of increasing the complication of the results.

If we take account of spin using the formalism of the previous section, formula (4.4) is replaced by

$$-G^2 \frac{(\gamma \cdot Q + m + M)[Q_\lambda Q_\mu - K_\lambda K_\mu - (Q_\lambda K_\mu - K_\lambda Q_\mu)]}{(s + t - 2M^2 - 2i\Delta - 2\mu^2 + m^2)}. \tag{4.12}$$

We have calculated the contribution of this term to the various helicity amplitudes $\zeta^J(\lambda, \lambda')$. To give an idea of the sizes we quote in Table I the values of their imaginary parts at $s = 120\mu^2$ which would be the position of the peak in the model neglecting spin discussed above. Note that the numbers quoted must be multiplied by isotopic spin factors of $\frac{1}{3}$, $-\frac{2}{3}$, and 1 in the $T = \frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$ states respectively (see Appendix B[2]): The signs quoted are such that a positive sign corresponds to an enhancement in the isobar production amplitude (i.e., positive r).

The simplest contribution to the amplitude for $\pi + N \to \pi + N^*$ is the pole contribution of Fig. 10. This contributes a term of the form

[2] The extra factors of 2 (process II) and $\sqrt{6}$ (process I) come from the differences between the coupling constants for different charge states: g is defined as the coupling constant of a neutral pion with a nucleon, while G is the coupling constant for a positive pion and proton to form an isobar.

TABLE I

SINGLE-NUCLEON POLE CONTRIBUTIONS TO PROCESS (II)

Contribution of Im(pole) to	$J = \frac{1}{2}+$	$J = \frac{1}{2}-$	$J = \frac{3}{2}+$	$J = \frac{3}{2}-$	$J = \frac{5}{2}+$	$J = \frac{5}{2}-$
$\zeta^{J\pm}(\frac{1}{2}, \frac{1}{2})$	0.95	−0.48	−0.11	0.40	0.45	0.01
$\zeta^{J\pm}(\frac{1}{2}, \frac{3}{2})$	—	—	−0.05	0.35	0.31	0.00
$\zeta^{J\pm}(\frac{3}{2}, \frac{3}{2})$	—	—	−0.46	−0.02	0.32	0.00

$$-gG\,\frac{\gamma_5(\gamma\cdot Q + (M+m)/2)(Q_\lambda + K_\lambda)}{(s + t - M^2 - i\Delta - 2\mu^2)} \tag{4.13}$$

which is smoothly varying in the physical region. The contributions of this term to the helicity amplitudes are given in Table II. The isotopic spin factors are in this case $2\sqrt{2/3}$ for $T = \frac{1}{2}$ and $-\sqrt{10}$ for $T = \frac{3}{2}$ (see Appendix B[2]).

V. CONCLUSIONS

The discussion of Section IV shows that if we consider the scattering of scalar pions from spinless isobars (formed from scalar pions and spinless nucleons) there is indeed a singularity in the upper half energy plane on the physical sheet for the partial wave projection of the process of Fig. 9 which should produce a maximum in this elastic cross section. We have further written down an amplitude which has a singularity in the appropriate region and which, for values of the energy near the maximum, satisfies an approximate unitarity condition in which intermediate states of pion plus nucleon and pion–pion isobar plus nucleon

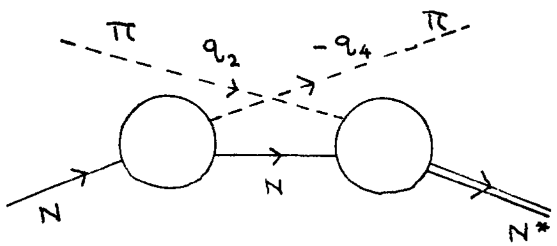

FIG. 10. Single-nucleon pole contribution to process (I).

TABLE II

SINGLE-NUCLEON POLE CONTRIBUTIONS TO PROCESS (I)

Contribution of | pole | to	$J = \frac{1}{2}+$	$J = \frac{1}{2}-$	$J = \frac{3}{2}+$	$J = \frac{3}{2}-$	$J = \frac{5}{2}+$	$J = \frac{5}{2}-$
$\phi^{J\pm}(\frac{1}{2}, \frac{1}{2})$	0.62	0.21	0.20	0.07	0.02	0.05
$\phi^{J\pm}(\frac{1}{2}, \frac{3}{2})$	—	—	0.35	0.07	0.03	0.15

are neglected. This takes the form of multiplying the expression (4.2) describing the process of Fig. 9 by a denominator which enhances or suppresses the maximum depending on the sign of the residue in (4.2). When we consider the full problem of the actual pions and nucleon with its full spin dependence we find expressions for the term analogous to (4.2) for the eigenstates of angular momentum, parity and isotopic spin and observe that for the $T = \frac{1}{2}$, $J = \frac{3}{2}$ state the residue is positive, which would indicate an enhancement in the simplified problem. Enhancements should also be expected in a number of other states. If we make the same approximations in the unitarity condition and consider the inelastic scattering into the pion plus nucleon channel, then there appears a "resonance" corresponding to the elastic maximum, but it is quite small in its effect. In view of the rather drastic approximation made for the unitarity condition, we cannot draw any firm conclusions about the validity of the mechanism as an explanation of the second resonances. However it is clear that with the formalism of this paper it should be possible to attempt a more accurate solution of this problem. In particular it is interesting to note that for the $T = \frac{5}{2}$ state, the unitarity condition we have used should be exact in the neighborhood of the maximum. Thus even a limited extension of our present calculations may be susceptible to experimental test.

APPENDIX A. CALCULATION OF HELICITY AMPLITUDES

Relations (3.21) for the four helicity amplitudes for process (I) may be obtained as follows: Define operators

$$\mathcal{L}(q_1) = \frac{\gamma \cdot q_1 + m}{\sqrt{2m(E_q + m)}}; \qquad \mathcal{L}(-q_3) = \frac{-\gamma \cdot q_3 + M}{\sqrt{2m(E_q' + M)}} \tag{A.1}$$

$$\mathcal{R} = \cos \theta/2 + i\sigma_2 \sin \theta/2$$

in spinor space, and

$$\mathbf{L} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & E_q'/M & -iq/M \\ & & iq/M & E'/M \end{pmatrix} \tag{A.2}$$

$$\mathbf{R} = \begin{pmatrix} \cos \theta & & \sin \theta & \\ & 1 & & \\ -\sin \theta & & \cos \theta & \\ & & & 1 \end{pmatrix} \tag{A.3}$$

in four-vector space. (We use a positive definite metric with four-vector $(i\mathbf{a}, a_0)$. Then the leading (2×2) minor in spinor space and the "space part" in four-vector space, $\mathfrak{M}'$, of

$$[\mathcal{L}(-q_3)\mathcal{R}][\mathbf{RL}]^T\mathfrak{M}[\mathcal{L}(q_1)] \tag{A.4}$$

give the elements of the T-matrix between a Pauli spinor on the right and a nonrelativistic spin $- \frac{3}{2}$ wave function in the uncoupled representation

$$\mathbf{1} + \mathbf{1/2} = \mathbf{3/2} \tag{A.5}$$

on the left. The initial and final wave functions are quantized along the directions of q_1 and $-q_2$ respectively. The helicity matrix is then given by

$$\Phi = \frac{i\sqrt{mM}}{4\pi W} (\mathbf{BA})^{+}\mathfrak{M}' \tag{A.6}$$

where

$$\mathbf{B} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \tag{A.7}$$

and A is just a matrix of the vector coupling coefficients of (A.5):

$$\mathbf{A} = \begin{pmatrix} \alpha & \frac{1}{\sqrt{3}}\beta & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}}\alpha & \sqrt{\frac{2}{3}}\beta & 0 \\ 0 & 0 & \sqrt{\frac{1}{3}}\alpha & \beta \end{pmatrix} \tag{A.8}$$

with $\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Carrying out the matrix multiplication of (A.4), (A.6) we get

$$\begin{aligned} \phi_1 \pm \phi_4 &= \frac{qs}{2\sqrt{2}} (\bar{c}G_1^{+} \mp \bar{s}G_2^{+}) \\ \phi_2 \pm \phi_3 &= \frac{-qs}{2\sqrt{6}} (\bar{c}G_1^{+} \mp \bar{s}G_2^{+}) + \frac{1}{\sqrt{6}}\left(\frac{E_q'}{M} qC + \frac{\omega_q}{M}\right)(\bar{c}G_1^{+} \pm \bar{s}G_2^{+}) \\ &\quad + \frac{1}{\sqrt{6}} \frac{q'W}{M} (\bar{c}G_1^{-} \pm \bar{s}G_2^{-}) \end{aligned} \tag{A.9}$$

where $G_1^{\pm}$, $G_2^{\pm}$ are expressed in terms of the invariant amplitudes by (3.17), (3.18). Equation (A.9), manipulating the rotation matrices of (3.15) in terms of Legendre polynomials, yields (3.21).

In the same way, we can define the matrix

$$[\mathfrak{L}(-p_3)\mathfrak{R}][\mathbf{RL}]^{T}\mathfrak{M}[L][\mathfrak{L}(p_1)] \tag{A.10}$$

for process (II). The leading (2×2) minor in spinor space and the "space-space" submatrix in four-vector space, $\mathfrak{M}'$ of (A.10) gives the T-matrix elements between nonrelativistic spin $-\frac{3}{2}$ wave functions in the uncoupled representation. We then have:

$$\zeta = \frac{M}{4\pi W} (\mathbf{BA})^{+}\mathfrak{M}'(\mathbf{BA}) \tag{A.11}$$

which gives:

$$\begin{aligned}
\zeta_1 \pm \zeta_4 = \frac{1}{\sqrt{3}} \Big\{ & \left(\frac{E_p}{M} S \pm \frac{1}{2}\right) (\bar{c}H_1 \pm \bar{s}K_1) \\
& - \frac{p^2}{2M} S \left[\frac{1}{2} E_p(1 + C) + \omega_p\right] (\bar{c}H_2 \pm \bar{s}K_2) \\
& - \frac{E_p}{4M} p^2 S(1 - C)(\bar{c}H_3 \pm \bar{s}K_3) + \frac{1}{2} C(\bar{s}K_1 \mp \bar{c}H_1) \\
& + \frac{1}{8} p^2 S^2[\bar{s}(K_2 - K_3) \mp \bar{c}(H_2 - H_3)] \Big\} \\
\zeta_2 \pm \zeta_3 = \frac{1}{3} \Big\{ & \frac{1}{2} (\bar{c}H_1 \pm \bar{s}K_1) + \left[\frac{1}{2} C + \frac{1}{M^2} (2E_p^2 C - p^2)\right] (\bar{c}H_1 \mp \bar{s}K_1) \\
& \cdot \left[\frac{1}{8} p^2 S^2 - \frac{1}{2} \frac{E_p^2}{M^2} p^2 (1 + C^2) - \frac{2E_p \omega_p}{M^2} p^2 - \frac{2\omega_p^2}{M^2} p^2\right] (\bar{c}H_2 \mp \bar{s}K_2) \\
& - \frac{p^2}{M^2} \left(E_p C \pm \frac{M}{2} S\right) (E_p + 2\omega_p)(\bar{c}H_2 \pm \bar{s}K_2) \\
& - \frac{1}{2} p^2 \left[\frac{1}{4} S^2 - \frac{E_p^2}{M^2} (1 + C^2)\right] (\bar{c}H_3 \mp \bar{s}K_3) \\
& - \frac{p^2}{M^2} E_p^2 C(\bar{c}H_3 \pm \bar{s}K_3) - \frac{2E_p}{M} S(\bar{s}K_1 \pm \bar{c}H_1) \\
& + \frac{1}{2} \frac{E_p}{M} p^2 SC(\bar{s}K_2 \mp \bar{c}H_2) + \frac{1}{2} \frac{E_p}{M} p^2 S(\bar{s}K_3 \mp \bar{c}H_3) \\
& + \frac{1}{2} \frac{E_p}{M} p^2 S(\bar{s}K_3 \mp \bar{c}H_3) \Big\} \\
\zeta_5 \pm \zeta_6 = {} & \frac{1}{2} (\bar{c}H_1 \pm \bar{s}K_1) + \frac{1}{2} C(\bar{c}H_1 \mp \bar{s}K_1) \\
& + \frac{1}{8} p^2 S^2[\bar{c}(H_2 - H_3) \pm \bar{s}(K_2 - K_3)]
\end{aligned} \tag{A.12}$$

where $\zeta_1 \cdots \zeta_4$ are defined in analogy with Eq. (3.15) and ζ_5, ζ_6 by

$$\begin{aligned}
\zeta_5 &= \langle \tfrac{3}{2} \mid \zeta \mid \tfrac{3}{2} \rangle = \frac{1}{2q} \sum_J (2J + 1) \zeta_5^J \, d^J_{3/2\,3/2}(\theta) \\
\zeta_6 &= \langle -\tfrac{3}{2} \mid \zeta \mid \tfrac{3}{2} \rangle = \frac{1}{2q} \sum_J (2J + 1) \zeta_6^J \, d^J_{3/2\,-3/2}(\theta).
\end{aligned} \tag{A.13}$$

In Eq. (A.13), we define H_i, K_i by

$$\begin{aligned} H_i &= \frac{M}{4\pi W}\left(c_i + \frac{W^2 - M^2 - \mu^2}{2M} D_i\right) \\ K_i &= \frac{M}{4\pi W}\left(\frac{E_p}{M} c_i + \omega D_i\right) \end{aligned} \tag{A.14}$$

and s, c, $\bar{s}$, $\bar{c}$ now refer to the angle of scattering in process (II). The combinations of definite parity are

$$\begin{aligned} \zeta^{J\pm}(\tfrac{1}{2}\,\tfrac{3}{2}) &= \zeta_1{}^J \pm \zeta_4{}^J \\ \zeta^{J\pm}(\tfrac{1}{2}\,\tfrac{1}{2}) &= \zeta_2{}^J \pm \zeta_3{}^J \\ \zeta^{J\pm}(\tfrac{3}{2}\,\tfrac{3}{2}) &= \zeta_5{}^J \pm \zeta_6{}^J \end{aligned} \tag{A.15}$$

where the (+) sign is associated with parity $(-1)^{J+1/2}$. Then, defining the combinations of Legendre polynomials

$$\begin{aligned} X_n &= \frac{P_{n+2} - P_n}{2n+3} \\ Y_n &= \frac{(n+2)P_{n+2} + (n+1)P_n}{2n+3} \\ Z_n &= (n+2)Y_n + nY_{(n-1)}, \end{aligned} \tag{A.16}$$

the formulas for the helicity amplitudes of Eq. (A.15) may conveniently be expressed in terms of the integrals

$$\begin{aligned} P_{1n}^{\pm} &= \int_{-1}^{1} (H_1 \pm (-1)^{n-l} K_1) P_n \, dC \\ P_{2n}^{\pm} &= p^2 \int_{-1}^{1} (H_2 \pm (-1)^{n-l} K_2) P_n \, dC \\ P_{3n}^{\pm} &= p^2 \int_{-1}^{1} [(H_2 - H_3) \pm (-1)^{n-l}(K_2 - K_3)] P_n \, dC. \end{aligned} \tag{A.17}$$

where $l = J - \tfrac{1}{2}$ and the integrals X_{in}, Y_{in}, Z_{in} $(i = 1, 2, 3)$ are obtained by substituting

$$P_n \rightarrow X_n, Y_n, Z_n$$

in Eq. (A.17). The formulas are

$$\begin{aligned} \zeta^{J+}(\tfrac{1}{2}, \tfrac{3}{2}) = p\sqrt{\frac{l(l+2)}{3}} \Bigg\{ & \frac{1}{4}(X_{1l}^{-} - X_{1(l-1)}^{-}) - \frac{E_p}{2M}(X_{1l}^{-} + X_{1(l-1)}^{-}) \\ & - \frac{E_p}{8M}(X_{3l}^{-} + X_{3(l-1)}^{-}) + \frac{E_p}{8M}\left(\frac{X_{2l}^{+} - X_{2(l-2)}^{+}}{2l+1} + \frac{X_{2(l+1)}^{+} - X_{2(l-1)}^{+}}{2l+3}\right) \\ & + \frac{1}{16}\left(\frac{X_{2l}^{-} - X_{2(l-2)}^{-}}{2l+1} + \frac{X_{2(l+1)}^{-} - X_{2(l-1)}^{-}}{2l+3}\right) - \frac{1}{16}(X_{3l}^{+} - X_{3(l-1)}^{+}) \Bigg\} \end{aligned}$$

$$\begin{aligned}
\zeta^{J+}(\tfrac{1}{2},\tfrac{1}{2}) = \frac{p}{3}\Big\{&\frac{1}{4}(P_{1l}^- + P_{1(l+1)}^-) - \frac{p^2}{M^2}(P_{1l}^+ + P_{1(l+1)}^+) \\
&- \frac{E_p^2}{2M^2}(P_{3l}^+ + P_{3(l+1)}^+) + \left(\frac{1}{4} + \frac{E_p^2}{M^2}\right)(Y_{1l}^- + Y_{1(l-1)}^-) \\
&+ \frac{E_p^2}{2M^2}(Y_{3l}^- + Y_{3(l-1)}^-) - \frac{EW}{M^2}(Y_{2l}^- + Y_{2(l-1)}^-) \\
&- \frac{1}{4}\left(\frac{1}{4} + \frac{E_p^2}{M^2}\right) Z_{3l}^+ - \frac{1}{4}\frac{E_p}{M} Z_{3l}^- - \frac{E_p}{M} Z_{1l}^- - \frac{W}{2M} Z_{2l}^- \\
&- \frac{1}{4}\left(\frac{1}{4} + \frac{E_p^2}{M^2}\right)\Big[\frac{l+2}{2l+3}(Y_{3(l+1)}^+ - Y_{3(l-1)}^+) \\
&- \frac{l}{2l+1}(Y_{3l}^+ - Y_{3(l-2)}^+)\Big] \\
&+ \frac{E_p}{M}\left[\frac{l+2}{2l+3}(Y_{3(l+1)}^+ - Y_{3(l-1)}^+) + \frac{l}{2l+1}(Y_{3l}^+ - Y_{3(l-2)}^+)\right]
\end{aligned} \tag{A.18}$$

$$\begin{aligned}
\zeta^{J+}(\tfrac{3}{2},\tfrac{3}{2}) = p\{&\tfrac{1}{2}[-X_{1l}^- + \tfrac{1}{2} Y_{1l}^- + \tfrac{1}{2}P_{1l}^-) \\
&- (X_{1(l-1)}^- - Y_{1(l-1)}^- + P_{1(l+1)}^-)] \\
&- \frac{1}{16}\left[\frac{(l-1)Y_{3l}^+ + (l+2)Y_{3(l-2)}^+}{2l+1} + \frac{lY_{3(l+1)}^+ + (l+3)Y_{3(l-1)}^+}{2l+3}\right]\Big\}
\end{aligned}$$

and the other three equations follow from the rule:

$$\text{Under}\qquad W \to -W, \qquad \begin{aligned}
&\zeta^{J+}(\tfrac{1}{2}\,\tfrac{3}{2}) \to -\zeta^{J-}(\tfrac{1}{2}\,\tfrac{3}{2}) \\
&\zeta^{J+}(\tfrac{1}{2}\,\tfrac{1}{2}) \to \zeta^{J-}(\tfrac{1}{2}\,\tfrac{1}{2}) \\
&\zeta^{J+}(\tfrac{3}{2}\,\tfrac{3}{2}) \to \zeta^{J-}(\tfrac{3}{2}\,\tfrac{3}{2}).
\end{aligned} \tag{A.19}$$

Finally, we quote the relations between the ϕ_i^J, ζ_i^J and the elements of the T-matrix between states of definite orbital angular momentum. These are, with $l = J - \tfrac{1}{2}$,

$$\begin{aligned}
S_{ll}^J &= -\sqrt{\frac{3(l+2)q'}{4(J+1)q}}\,\phi^{J-}(\tfrac{1}{2}\,\tfrac{3}{2}) - \sqrt{\frac{lq'}{4(J+1)q}}\,\phi^{J-}(\tfrac{1}{2}\,\tfrac{1}{2}) \\
S_{(l+1)(l+1)}^J &= -\sqrt{\frac{3lq'}{4Jq}}\,\phi^{J+}(\tfrac{1}{2}\,\tfrac{3}{2}) + \sqrt{\frac{(l+2)q'}{4Jq}}\,\phi^{J+}(\tfrac{1}{2}\,\tfrac{1}{2}) \\
S_{l(l+2)}^J &= -\sqrt{\frac{lq'}{4(J+1)q}}\,\phi^{J-}(\tfrac{1}{2}\,\tfrac{3}{2}) + \sqrt{\frac{3(l+2)q'}{4(J+1)q}}\,\phi^{J-}(\tfrac{1}{2}\,\tfrac{1}{2})
\end{aligned} \tag{A.20}$$

$$S^J_{(l+1)(l-1)} = -\sqrt{\frac{(l+2)q'}{4Jq}}\,\phi^{J+}(\tfrac{1}{2}\,\tfrac{3}{2}) - \sqrt{\frac{3lq'}{4Jq}}\,\phi^{J+}(\tfrac{1}{2}\,\tfrac{1}{2})$$

for process (I), and

$$\begin{aligned}
T^J_{l(l+2)} = T^J_{(l+2)l} &= -\frac{\sqrt{3l(l+2)}}{4(J+1)}\,\zeta^{J-}(\tfrac{1}{2}\,\tfrac{1}{2}) \\
&\quad - \frac{l+3}{2(J+1)}\,\zeta^{J-}(\tfrac{1}{2}\,\tfrac{3}{2}) + \frac{\sqrt{3l(l+2)}}{4(J+1)}\,\zeta^{J-}(\tfrac{3}{2}\,\tfrac{3}{2}) \\
T^J_{(l-1)(l+1)} = T^J_{(l+1)(l-1)} &= -\frac{\sqrt{3l(l+2)}}{4J}\,\zeta^{J+}(\tfrac{1}{2}\,\tfrac{1}{2}) \\
&\quad + \frac{l-1}{2J}\,\zeta^{J+}(\tfrac{1}{2}\,\tfrac{3}{2}) + \frac{\sqrt{3l(l+2)}}{4J}\,\zeta^{J+}(\tfrac{3}{2}\,\tfrac{3}{2}) \\
T^J_{(l-1)(l-1)} &= \frac{3l}{4J}\,\zeta^{J+}(\tfrac{1}{2}\,\tfrac{1}{2}) + \frac{\sqrt{3l(l+2)}}{2J}\,\zeta^{J+}(\tfrac{1}{2}\,\tfrac{3}{2}) \\
&\quad + \frac{l+2}{4J}\,\zeta^{J+}(\tfrac{3}{2}\,\tfrac{3}{2}) \\
T^J_{(l+2)(l+2)} &= \frac{3(l+2)}{4(J+1)}\,\zeta^{J-}(\tfrac{1}{2}\,\tfrac{1}{2}) - \frac{\sqrt{3l(l+2)}}{2(J+1)}\,\zeta^{J-}(\tfrac{1}{2}\,\tfrac{3}{2}) \\
&\quad + \frac{l}{4(J+1)}\,\zeta^{J-}(\tfrac{3}{2}\,\tfrac{3}{2}) \\
T^J_{ll} &= \frac{l}{4(J+1)}\,\zeta^{J-}(\tfrac{1}{2}\,\tfrac{1}{2}) + \frac{\sqrt{3l(l+2)}}{2(J+1)}\,\zeta^{J-}(\tfrac{1}{2}\,\tfrac{3}{2}) \\
&\quad + \frac{3(l+2)}{4(J+1)}\,\zeta^{J-}(\tfrac{3}{2}\,\tfrac{3}{2}) \\
T^J_{(l+1)(l+1)} &= \frac{l+2}{4J}\,\zeta^{J+}(\tfrac{1}{2}\,\tfrac{1}{2}) - \frac{\sqrt{3l(l+2)}}{2J}\,\zeta^{J+}(\tfrac{1}{2}\,\tfrac{3}{2}) \\
&\quad + \frac{3l}{4(J+1)}\,\zeta^{J+}(\tfrac{3}{2}\,\tfrac{3}{2})
\end{aligned} \tag{A.21}$$

for process (II).

APPENDIX B. ISOTOPIC SPIN CROSSING RELATIONS

Consider some scattering reaction

$$A + B \to C + D \tag{B.1}$$

described by a transition matrix $\mathfrak{M}$, and the corresponding crossed reaction

$$\bar{C} + B \to \bar{A} + D \tag{B.2}$$

described by a transition matrix $\mathfrak{M}$. In this appendix we shall be concerned solely with the dependence on isotopic spin, which we shall call simply spin, as there is no ambiguity.

Let the spins of particles A, B, C, D be a, b, c, d and let $\alpha, \beta, \gamma, \delta$ be the z-components. We shall use the convention that a denotes spin a, z-component α, while $\bar{a}$ denotes spin a, z-component $-\alpha$, and omit the explicit z-components wherever possible. The matrix element of M corresponding to a particular charge state may be expanded in terms of matrix elements corresponding to given total spin e (z-component ε):

$$\langle\alpha\beta \mid \mathfrak{M} \mid \gamma\delta\rangle = \sum_{e\varepsilon} \langle ab \mid e\rangle\langle e \mid cd\rangle M_e \tag{B.3}$$

where $M_e = \langle e\varepsilon \mid \mathfrak{M} \mid e\varepsilon\rangle$ is independent of ε by charge independence.

Similarly

$$\langle -\gamma\beta \mid \mathfrak{N} \mid -\alpha\delta\rangle = \sum_{f\phi} \langle \bar{c}b \mid f\rangle\langle f \mid \bar{a}d\rangle N_f \tag{B.4}$$

where f, ϕ are the total spin and z-component for reaction (2). We note that

$$\langle ab \mid e\rangle = \langle e \mid ab\rangle \equiv \mathbf{C}(abe; \alpha\beta\varepsilon) \quad \text{etc.} \tag{B.5}$$

are simply Clebsch-Gordan coefficients.

Now we introduce the crossing assumption which is that if A and C are identical particles (implying $a = c$)

$$\langle\alpha\beta \mid \mathfrak{M} \mid \gamma\delta\rangle = (-1)^{\alpha+\gamma}\langle -\gamma\beta \mid \mathfrak{N} \mid -\alpha\delta\rangle \tag{B.6}$$

provided that corresponding momentum and angular momentum states are considered. The factor $(-1)^{\alpha+\gamma}$ comes from the behavior of spin states under time-reversal. Next we use the result

$$\sum_{\varepsilon} \langle ab \mid e\rangle\langle cd \mid e\rangle = (2e+1)\sum_{f\phi} (-1)^{2f-b-d-\alpha-\gamma}\mathbf{W}(abdc; ef)\langle \bar{c}b \mid f\rangle\langle \bar{a}d \mid f\rangle \tag{B.7}$$

where $\mathbf{W}(\quad)$ is the Racah coefficient. Now inserting (7) into (3) and then (3) and (4) into (6) we immediately obtain

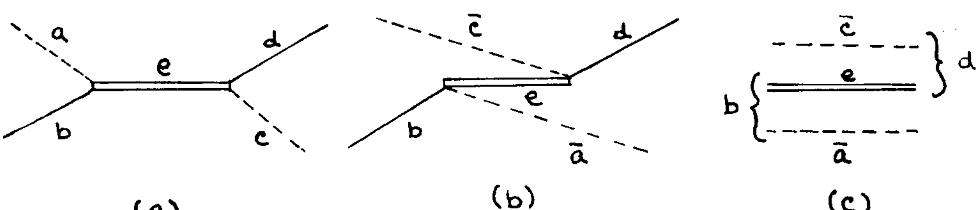

FIG. 11. Diagrams corresponding to process (B.1) and (B.2).

$$N_f = (-1)^{2f-b-d} \sum_e (2e+1) \mathbf{W}(abdc; ef) M_e$$
$$= \sum_e (-1)^{e-f} (2e+1) \mathbf{W}(aefc; bd) M_e = \sum_e \mathbf{X}_{ef} M_e, \tag{B.8}$$
$$\mathbf{X}_{ef} = (-1)^{e-f} (2e+1) \mathbf{W}(aefc; bd)$$

where $\mathbf{X}_{ef}$ is the desired crossing matrix.

The occurrence of the Racah coefficient in this result can easily be understood if we consider the amplitude M_e to be described by the diagram of Fig. 11(a). The corresponding crossed amplitude is represented by (b). $\mathbf{X}_{ef}$ is then the contribution of diagram (b) to the state of total spin f. This is simply the overlap between the two ways of combining the three particle intermediate state $(\bar{a}\ e\ \bar{c})$ to total spin f, with either $\bar{a} + e \rightarrow b$ (initial state) or $\bar{c} + e \rightarrow d$ (final state) as indicated in Fig. 11(c). But this is essentially the definition of the Racah coefficient $\mathbf{W}(aefc;\ bd)$ which is equal to $\mathbf{X}_{ef}$ apart from trivial factors.

The two cases we are concerned with are

(1) $\pi N \rightarrow \pi N^*$. $a = c = 1$; $b = \frac{1}{2}, d = \frac{3}{2}$; $e, f = \frac{1}{2}, \frac{3}{2}$

$$\mathbf{X}_{ef} = \frac{1}{6} \begin{pmatrix} 4 & -\sqrt{10} \\ -2\sqrt{10} & -4 \end{pmatrix} \tag{B.9}$$

(2) $\pi N^* \rightarrow \pi N^*$. $a = c = 1$; $b = d = \frac{3}{2}$; $e, f = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$.

$$\mathbf{X}_{ef} = \frac{1}{30} \begin{pmatrix} 5 & -10 & 15 \\ -20 & 22 & 12 \\ 45 & 18 & 3 \end{pmatrix} \tag{B.10}$$

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